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Multiple Positive Solutions of Nonlinear Two-Point Boundary Value Problems*

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We improve the results obtained by Erbe, Hu, and Wang in a recent paper. We show that there exist at least two positive solutions of two-point boundary value problems under conditions weaker than those used by Erbe, Hu, and Wang.

1. INTRODUCTION

In a recent paper [2], Erbe, Hu, and Wang studied the existence of multiple positive solutions of the following second order two-point boundary value problems (BVP),

$$-u'' = f(t,u), \qquad 0 < t < 1, \tag{1}$$

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{2}$$

where $f \in C(I \times R_+, R_+)$, $I = [0,1]$, $R_+ = [0, +\infty)$, $\alpha, \beta, \gamma, \delta \geq 0$, and $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. They proved that there exist at least two positive solutions of (1)–(2) under conditions (H1) and (H3) (or under conditions (H2) and (H4)), which are listed as follows:

(H1) $\lim_{u \to 0+} \min_{t \in [0,1]}(f(t,u)/u) = \infty$, $\lim_{u \to +\infty} \min_{t \in [0,1]} (f(t,u)/u) = \infty$;

(H2) $\lim_{u \to 0+} \max_{t \in [0,1]}(f(t,u)/u) = 0$, $\lim_{u \to +\infty} \max_{t \in [0,1]}(f(t,u)/u) = 0$;

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(H3) There is a $p > 0$ such that $0 \le u \le p$ and $0 \le t \le 1$ implies

$$f(t, u) \le \eta p,$$

where

$$\eta = \left(\int_0^1 G(s, s)\, ds \right)^{-1} = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta}$$

and $G(t, s)$ is the Green's function to $-u'' = 0$ subject to the boundary condition (2);

(H4) There is a $p > 0$ such that $\sigma p \le u \le p$ implies

$$f(t, u) \ge \lambda p,$$

where $\lambda^{-1} = \int_{1/4}^{3/4} G(1/2, s)\, ds$, and

$$\sigma = \min\left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{\alpha(4 + \beta)} \right\}.$$

In this paper, we prove that (1)–(2) possess at least two positive solutions under weaker conditions than those listed above. By the way, a small error in the proof of Theorem 3 in paper [2] is corrected. We denote the first eigenvalue of $-u'' = \lambda u$ subject to the boundary condition (2) by λ_1, and the corresponding eigenfunction by $\phi_1(t)$. It is well known that $\lambda_1 > 0$ and $\phi_1(t)$ does not change sign in $(0, 1)$, and therefore, without loss of generality, we can assume that $\phi_1(t) > 0$ for $0 < t < 1$ and $\|\phi_1\| = \max_{0 \le t \le 1} |\phi_1(t)| = 1$. We formulate some conditions for $f(t, u)$ as follows which will play roles in this paper.

(H1)* $\liminf_{u \to 0+} \min_{t \in [0,1]} (f(t, u)/u) > \lambda_1$, $\liminf_{u \to +\infty} \min_{t \in [0,1]} (f(t, u)/u) > \lambda_1$;

(H2)* $\limsup_{u \to 0+} \max_{t \in [0,1]} (f(t, u)/u) < \lambda_1$, $\limsup_{u \to +\infty} \max_{t \in [0,1]} (f(t, u)/u) < \lambda_1$;

(H3)* There is a $p > 0$ such that $0 \le u \le p$ and $0 \le t \le 1$ implies

$$f(t, u) < \eta^* p,$$

where

$$\eta^* = \frac{8\rho^2}{(\gamma + 2\delta)(4\beta\rho + \alpha^2\gamma + 2\alpha^2\delta)}.$$

(P) For any $r > 0$, there is a $M_r > 0$ such that $0 \le t \le 1$ and $0 \le u_1 \le u_2 \le r$ implies

$$f(t, u_2) - f(t, u_1) \ge -M_r(u_2 - u_1).$$

2. PRELIMINARIES AND LEMMAS

We adopt the symbols used in [2]. By Lemma 2 in [2], finding positive solutions of (1)–(2) is equivalent to finding nontrivial fixed points of F: $K \to K$, where

$$K = \left\{ u \in X | u(t) \ge 0 \text{ for all } t \in [0,1] \text{ and } \min_{1/4 \le t \le 3/4} u(t) \ge \sigma \|u\| \right\},$$

$X = C[0,1]$, and F: $K \to K$ given by

$$Fu(t) = \int_0^1 G(t,s) f(s, u(s))\, ds,$$

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \le t \le s \le 1. \end{cases}$$

For $r > 0$, the set $\{u \in K | \, \|u\| < r\}$ is denoted by K_r.

Now we mention that the claim, i.e., $Fu \ne u$ for $u \in \partial K_p$ in the proof of Theorem 3 in paper [2], may not be true under the condition (H3). In fact, if $\alpha = \delta = 1$, $\beta = \gamma = 0$, then it is easy to see that $\rho = 1$, $\eta = 2$, and (H3) reduces to the condition that there is a $p > 0$ such that $0 \le u \le p$ and $0 \le t \le 1$ implies $f(t,u) \le 2p$, which includes the case that $f(t,u) = 2p$ for any $0 \le u \le p$ and $0 \le t \le 1$. In this particular case, if we take $u_0(t) = -pt^2 + 2pt$, then $u_0(t)$ is a positive solution of (1)–(2) and $\|u_0\| = p$, and therefore $Fu_0 = u_0$, $u_0 \in \partial K_p$. Hence $Fu \ne u$ for $u \in \partial K_p$ may not be true under the condition (H3). Checking the proof of Theorem 3 in [2], we find that the error can be corrected by replacing (H3) with (H3)$'$.

(H3)$'$ There is a $p > 0$ such that $0 \le u \le p$ and $0 \le t \le 1$ implies

$$f(t,u) < \eta p,$$

where η is just as in (H3).

We recall some lemmas which will be used in this paper. First of all, according to the proofs of Theorem 3 and Theorem 4 in [2], the following three lemmas are valid.

LEMMA 1. *If* $(H1)$ *is satisfied, then there exist* $0 < r_0 < R_0 < \infty$ *such that* $i(F, K_r, K) = 0$ *for* $0 < r \le r_0$ *and* $i(F, K_R, K) = 0$ *for* $R \ge R_0$.

LEMMA 2. *If* (H3)′ *is satisfied, then* $i(F, K_p, K) = 1$.

LEMMA 3. *If* (H4) *is satisfied, then* $i(F, K_p, K) = 0$.

$\phi_0(t)$ is called a subsolution of the BVP (1)–(2) if $\phi_0(t) \in C^2[0,1]$, $\phi_0(t) \ge 0$, and it satisfies

$$-\phi_0''(t) \le f(t, \phi_0(t)), \qquad 0 < t < 1,$$

$$\begin{cases} \alpha\phi_0(0) - \beta\phi_0'(0) \le 0, \\ \gamma\phi_0(1) + \delta\phi_0'(1) \le 0, \end{cases}$$

and $\phi_0(t)$ is called a strict subsolution if it is a subsolution but it is not a solution. We call $\psi_0(t)$ a supersolution of the BVP (1)–(2) if $\psi_0(t) \in C^2[0,1]$, $\psi_0(t) \ge 0$, and it satisfies

$$-\psi_0''(t) \ge f(t, \psi_0(t)), \qquad 0 < t < 1,$$

$$\begin{cases} \alpha\psi_0(0) - \beta\psi_0'(0) \ge 0, \\ \gamma\psi_0(1) + \delta\psi_0'(1) \ge 0, \end{cases}$$

and $\psi_0(t)$ is called a strict supersolution if it is a supersolution but it is not a solution. Let $u, v \in X$. We use $u \le v$ to represent $u(t) \le v(t)$ for $t \in [0,1]$, and use $u < v$ to represent $u \le v$ but $u \ne v$.

LEMMA 4. *Assume that* (P) *is satisfied. If* $\phi_0(t)$ *and* $\psi_0(t)$ *are a strict subsolution and a strict supersolution of* (1)–(2), *respectively,* $\phi_0 < \psi_0$, *then* (1)–(2) *has a solution* $u^*(t)$ *satisfying* $\phi_0 < u^* < \psi_0$.

LEMMA 5. *Assume that there exists* $M > 0$ *such that*

$$f(t, u_2) - f(t, u_1) \ge -M(u_2 - u_1), \qquad \forall\, 0 \le t \le 1, 0 \le u_1 < u_2.$$

If $\phi_0(t)$ *and* $\psi_0(t)$ *are a strict subsolution and a strict supersolution of* (1)–(2), *respectively,* $\phi_0 \not\le \psi_0$, *and*

$$\psi_0(t) > 0, \qquad \forall\, 0 \le t \le 1,$$

then (1)–(2) *has a solution* $u^*(t)$ *satisfying* $\phi_0 \not\le u^* \not\le \psi_0$.

See [1, 3] for the proof of Lemma 4, and see [5, 6] for the proof of Lemma 5.

3. MAIN RESULTS

THEOREM 1. *If* (H1)* *and* (H3)′ *are satisfied, then the BVP* (1)–(2) *has at least two positive solutions* x_1 *and* x_2 *such that*

$$0 < \|x_1\| < p < \|x_2\|.$$

Proof. According to Lemma 2, we have that

$$i(F, K_p, K) = 1. \tag{3}$$

Fix $0 < m < 1 < n$, and let $f_1(u) = u^m + u^n$ for $u \geq 0$. Then $f_1(u)$ satisfies the condition (H1). Define F_1: $K \to K$ by

$$F_1 u(t) = \int_0^1 G(t,s) f_1(u(s))\, ds,$$

then by using Lemma 1, we conclude that there exist $0 < r_0 < p < R_0 < \infty$, such that $0 < r \leq r_0$ implies

$$i(F_1, K_r, K) = 0, \tag{4}$$

and $R \geq R_0$ implies

$$i(F_1, K_R, K) = 0. \tag{5}$$

Define H: $[0,1] \times K \to K$ by $H(s,u) = (1-s)Fu + sF_1 u$, then H is a completely continuous operator. By the first inequality in (H1)* and the defination of f_1, there are $\epsilon > 0$ and $0 < r_1 \leq r_0$ such that

$$f(t,u) \geq (\lambda_1 + \epsilon)u, \qquad \forall\, 0 \leq t \leq 1, 0 \leq u \leq r_1, \tag{6}$$

$$f_1(u) \geq (\lambda_1 + \epsilon)u, \qquad \forall\, 0 \leq u \leq r_1. \tag{7}$$

We now prove that $H(s,u) \neq u$ for all $0 \leq s \leq 1$ and $u \in \partial P_{r_1}$. In fact, if there exist $0 \leq s_0 \leq 1$ and $u_0 \in \partial P_{r_1}$ such that $H(s_0, u_0) = u_0$, then $u_0(t)$ satisfies the equation

$$-u_0''(t) = (1-s_0) f(t, u_0(t)) + s_0 f_1(u_0(t))$$

and the boundary condition (2). Multiplying the last equation by ϕ_1 and then integrating from 0 to 1, using integration by parts in the left hand side two times, we get that

$$\lambda_1 \int_0^1 u_0(t)\phi_1(t)\,dt = \int_0^1 \big((1-s_0) f(t, u_0(t)) + s_0 f_1(u_0(t))\big)\phi_1(t)\,dt, \tag{8}$$

which, combining with (6) and (7), yields that

$$\begin{aligned}\lambda_1 \int_0^1 u_0(t)\phi_1(t)\,dt &\geq \int_0^1 \big((1-s_0)(\lambda_1+\epsilon)u_0(t) \\ &\qquad + s_0(\lambda_1+\epsilon)u_0(t)\big)\phi_1(t)\,dt \\ &= (\lambda_1+\epsilon)\int_0^1 u_0(t)\phi_1(t)\,dt.\end{aligned}$$

Since

$$\int_0^1 u_0(t)\phi_1(t)\,dt > 0,$$

we see that $\lambda_1 \geq \lambda_1 + \epsilon$, which is a contradiction. By (4) and homotopy invariance of the fixed point index (cf. [3]), we get that

$$\begin{aligned} i(F, K_{r_1}, K) &= i(H(0,\cdot), K_{r_1}, K) \\ &= i(H(1,\cdot), K_{r_1}, K) \\ &= i(F_1, K_{r_1}, K) \\ &= 0. \end{aligned} \tag{9}$$

By the second inequality in (H1)* and the definition of f_1, there exist $\epsilon > 0$ and $M > 0$ such that

$$f(t,u) \geq (\lambda_1+\epsilon)u, \qquad \forall\, 0 \leq t \leq 1,\ u > M,$$
$$f_1(u) \geq (\lambda_1+\epsilon)u, \qquad \forall\, u > M.$$

Let

$$\begin{aligned} C = &\max_{0\leq t\leq 1,\, 0\leq u\leq M} |f(t,u) - (\lambda_1+\epsilon)u| \\ &+ \max_{0\leq u\leq M} |f_1(u) - (\lambda_1+\epsilon)u| + 1, \end{aligned}$$

then, it is obvious that

$$f(t,u) \geq (\lambda_1 + \epsilon)u - C, \qquad \forall\, 0 \leq t \leq 1,\, u \geq 0, \tag{10}$$

$$f_1(u) \geq (\lambda_1 + \epsilon)u - C, \qquad \forall\, u \geq 0. \tag{11}$$

We now prove that there exists $R_1 \geq R_0$ such that $H(s,u) \neq u$ for any $0 \leq s \leq 1$ and $u \in K$, $\|u\| \geq R_1$. In fact, if $0 \leq s_0 \leq 1$ and $u_0 \in K$ satisfy $H(s_0, u_0) = u_0$, then (8) is valid. Combine (8) and (10) and (11) to conclude that

$$\begin{aligned}\lambda_1 \int_0^1 u_0(t)\phi_1(t)\,dt &\geq \int_0^1 ((\lambda_1 + \epsilon)u_0(t) - C)\phi_1(t)\,dt \\ &= (\lambda_1 + \epsilon)\int_0^1 u_0(t)\phi_1(t)\,dt - C\int_0^1 \phi_1(t)\,dt,\end{aligned}$$

from which we see that

$$\int_0^1 u_0(t)\phi_1(t)\,dt \leq \frac{C}{\epsilon}\int_0^1 \phi_1(t)\,dt. \tag{12}$$

By maximum principles (cf. [4]), there exists exactly one $t_0 \in [0,1]$ such that $u_0'(t_0) = 0$, $u_0(t) \geq (t/t_0)\|u_0\|$ for $0 \leq t \leq t_0$ if $t_0 > 0$ and $u_0(t) \geq ((1 - t)/(1 - t_0))\|u_0\|$ for $t_0 \leq t \leq 1$ if $t_0 < 1$. When $u_0 \neq \theta$ and if $0 < t_0 < 1$, we have that

$$\begin{aligned}\int_0^1 u_0(t)\phi_1(t)\,dt &\geq \int_0^{t_0} \frac{t}{t_0}\|u_0\|\phi_1(t)\,dt + \int_{t_0}^1 \frac{1-t}{1-t_0}\|u_0\|\phi_1(t)\,dt \\ &> \|u_0\|\left(\int_0^{t_0} t\phi_1(t)\,dt + \int_{t_0}^1 (1-t)\phi_1(t)\,dt\right),\end{aligned}$$

and therefore,

$$\int_0^1 u_0(t)\phi_1(t)\,dt > \|u_0\|\int_0^1 t(1-t)\phi_1(t)\,dt. \tag{13}$$

If $t_0 = 0$ and $t_0 = 1$, (13) is still valid when $u_0 \neq \theta$. From (12) and (13) we see that if $0 \leq s_0 \leq 1$ and $u_0 \in K$ satisfy $H(s_0, u_0) = u_0$, then

$$\|u_0\| < \frac{C}{\epsilon}\left(\int_0^1 \phi_1(t)\,dt\right)\left(\int_0^1 t(1-t)\phi_1(t)\,dt\right)^{-1} =: \tilde{R}_1.$$

Let $R_1 = \max\{R_0, \tilde{R}_1\}$, then $H(s,u) \neq u$ for any $0 \leq s \leq 1$ and $u \in K$, $\|u\| \geq R_1$. By (5) and homotopy invariance of the fixed point index, we have that

$$\begin{aligned} i\big(F, K_{R_1}, K\big) &= i\big(H(0,\cdot), K_{R_1}, K\big) \\ &= i\big(H(1,\cdot), K_{R_1}, K\big) \\ &= i\big(F_1, K_{R_1}, K\big) \\ &= 0. \end{aligned} \tag{14}$$

Use (3), (9), and (14) to conclude that

$$i\Big(F, K_{R_1} \setminus \overline{K}_p, K\Big) = -1,$$

$$i\Big(F, K_p \setminus \overline{K}_{r_1}, K\Big) = 1.$$

Therefore F has fixed points x_1 and x_2 in $K_p \setminus \overline{K}_{r_1}$ and $K_{R_1} \setminus \overline{K}_p$, respectively, which means that $x_1(t)$ and $x_2(t)$ are positive solutions of (1)–(2) and $0 < \|x_1\| < p < \|x_2\|$. The proof is completed.

Remark 1. Theorem 1 improves Theorem 3 in [2].

THEOREM 2. *If* (H2)* *and* (H4) *are satisfied, then the BVP* (1)–(2) *has at least two positive solutions* x_1 *and* x_2 *such that*

$$0 < \|x_1\| < p < \|x_2\|.$$

Proof. According to Lemma 3, we have that

$$i\big(F, K_p, K\big) = 0. \tag{15}$$

Define H_1: $[0,1] \times K \to K$ by $H_1(s,u) = sFu$, then H_1 is a completely continuous operator. By the first inequality in (H2)*, there exist $\epsilon > 0$ and $0 < r_0 < p$ such that

$$f(t,u) \leq (\lambda_1 - \epsilon)u, \qquad \forall\, 0 \leq t \leq 1, 0 \leq u \leq r_0. \tag{16}$$

We now prove that $H_1(s,u) \neq u$ for $0 \leq s \leq 1$ and $u \in \partial P_{r_0}$. In fact, if there exist $0 \leq s_0 \leq 1$ and $u_0 \in \partial P_{r_0}$ such that $H_1(s_0, u_0) = u_0$, then the $u_0(t)$ satisfy the boundary condition (2) and

$$-u_0''(t) = s_0 f(t, u_0(t)), \qquad \forall\, 0 < t < 1.$$

Multiplying the last equality by $\phi_1(t)$ and integrating from 0 to 1, we see that

$$\lambda_1 \int_0^1 u_0(t)\phi_1(t)\,dt = s_0 \int_0^1 f(t, u_0(t))\phi_1(t)\,dt. \tag{17}$$

From (16) and (17), we get that

$$\lambda_1 \int_0^1 u_0(t)\,\phi_1(t)\,dt \le \int_0^1 f\big(t, u_0(t)\big)\,\phi_1(t)\,dt \le (\lambda_1 - \epsilon)\int_0^1 u_0(t)\,\phi_1(t)\,dt,$$

therefore, $\lambda_1 \le \lambda_1 - \epsilon$, which is a contradiction. Using homotopy invariance of the fixed point index, we have that

$$\begin{aligned} i\big(F, K_{r_0}, K\big) &= i\big(H_1(1,\cdot), K_{r_0}, K\big) \\ &= i\big(H_1(0,\cdot), K_{r_0}, K\big) \\ &= i\big(0, K_{r_0}, K\big) \\ &= 1. \end{aligned} \tag{18}$$

By the second inequality in (H2)*, there exist $\epsilon > 0$ and $M > 0$ such that

$$f(t,u) \le (\lambda_1 - \epsilon)u, \qquad \forall\, 0 \le t \le 1,\, u > M.$$

Set

$$C = \max_{0 \le t \le 1,\, 0 \le u \le M} \big| f(t,u) - (\lambda_1 - \epsilon)u \big| + 1,$$

then

$$f(t,u) \le (\lambda_1 - \epsilon)u + C, \qquad \forall\, 0 \le t \le 1,\, u \ge 0. \tag{19}$$

If $0 \le s_0 \le 1$ and $u_0 \in K$ satisfy $H_1(s_0, u_0) = u_0$, then (17) is valid. From (17) and (19) we see that

$$\begin{aligned} \lambda_1 \int_0^1 u_0(t)\,\phi_1(t)\,dt &\le \int_0^1 f\big(t, u_0(t)\big)\,\phi_1(t)\,dt \\ &\le (\lambda_1 - \epsilon)\int_0^1 u_0(t)\,\phi_1(t)\,dt + C\int_0^1 \phi_1(t)\,dt, \end{aligned}$$

and therefore,

$$\int_0^1 u_0(t)\,\phi_1(t)\,dt \le \frac{C}{\epsilon}\int_0^1 \phi_1(t)\,dt.$$

By the argument used in the proof of Theorem 1, there exist $R_0 > p$ such that $H_1(s, u) \neq u$ for $0 \leq s \leq 1$ and $u \in K$, $\|u\| \geq R_0$, and therefore

$$\begin{aligned} i\left(F, K_{R_0}, K\right) &= i\left(H_1(1, \cdot), K_{R_0}, K\right) \\ &= i\left(H_1(0, \cdot), K_{R_0}, K\right) \\ &= i\left(\theta, K_{R_0}, K\right) \\ &= 1. \end{aligned} \tag{20}$$

Combining (15), (18), and (20), we see that (1)–(2) has at least two positive solutions $x_1(t)$ and $x_2(t)$ satisfying $0 < \|x_1\| < p < \|x_2\|$. The proof is completed.

Remark 2. Theorem 2 improved Theorem 4 in [2].

THEOREM 3. *If* (H1)*, (H3)*, *and* (P) *are satisfied, then the BVP* (1)–(2) *has at least two positive solutions*.

Proof. By (H1)*, there exist $\epsilon > 0$ and $C > 0$ such that

$$f(t, u) \geq (\lambda_1 + \epsilon)u - C, \qquad \forall\, 0 \leq t \leq 1, u \geq 0. \tag{21}$$

From the proof of Theorem 1, there is a $\tilde{R}_1 > 0$, depending only on ϵ, C in (21) and $\phi_1(t)$, such that $\|u\| < \tilde{R}_1$ for any positive solution u of (1)–(2). Define $f_1(t, u)$ for $0 \leq t \leq 1$ and $u \geq 0$ as

$$f_1(t, u) = (1 - \mu(u))f(t, u) + (\lambda_1 + \epsilon)\mu(u)u,$$

where function $\mu\colon R_+ \to R_+$ is given by

$$\mu(u) = \begin{cases} 0, & \text{if } 0 \leq u \leq R_2, \\ u - R_2, & \text{if } R_2 < u < R_2 + 1,. \\ 1, & \text{if } u \geq R_2 + 1, \end{cases}$$

and $R_2 = \max\{\tilde{R}_1, p + 1\}$. For the same $\epsilon > 0$ and $C > 0$ as in (21), we have

$$f_1(t, u) \geq (\lambda_1 + \epsilon)u - C, \qquad \forall\, 0 \leq t \leq 1, u \geq 0,$$

from which we see that, for positive solution $u(t)$ of equation

$$-u''(t) = f_1(t, u(t)), \qquad 0 < t < 1 \tag{22}$$

subject to the boundary condition (2), it must be $\|u\| < \tilde{R}_1$, and therefore $u(t)$ is a positive solution of (1)–(2). Hence (1)–(2) is equivalent to (22)–(2)

under the condition (H1)*. We now claim that there exists $\tilde{M} > 0$ such that

$$f_1(t, u_2) - f_1(t, u_1) \geq -\tilde{M}(u_2 - u_1), \qquad \forall\, 0 \leq u_1 < u_2. \tag{23}$$

In fact, if $0 \leq u_1 < u_2 \leq R_2$, then by the condition (P),

$$f_1(t, u_2) - f_1(t, u_1) = f(t, u_2) - f(t, u_1) \geq -M_{R_2}(u_2 - u_1),$$

if $R_2 \leq u_1 \leq u_2 \leq R_2 + 1$, then by the definition of $f_1(t, u)$ and (P),

$$\begin{aligned} f_1(t, u_2) - f_1(t, u_1) &= (1 - \mu(u_2))f(t, u_2) + (\lambda_1 + \epsilon)\mu(u_2)u_2 \\ &\quad -(1 - \mu(u_1))f(t, u_1) - (\lambda_1 + \epsilon)\mu(u_1)u_1 \\ &= (1 - \mu(u_2))\big(f(t, u_2) - f(t, u_1)\big) \\ &\quad -(\mu(u_2) - \mu(u_1))f(t, u_1) \\ &\quad +(\lambda_1 + \epsilon)\mu(u_2)(u_2 - u_1) \\ &\quad +(\lambda_1 + \epsilon)(\mu(u_2) - \mu(u_1))u_1 \\ &\geq -M_{R_2+1}(u_2 - u_1) - C_1(u_2 - u_1) \\ &= -\big(M_{R_2+1} + C_1\big)(u_2 - u_1), \end{aligned}$$

where $C_1 = \max_{0 \leq t \leq 1,\, R_2 \leq u \leq R_2+1} |f(t, u)|$, and if $R_2 + 1 \leq u_1 < u_2$, then

$$f_1(t, u_2) - f_1(t, u_1) = (\lambda_1 + \epsilon)(u_2 - u_1).$$

Taking $\tilde{M} = \max\{M_{R_2}, M_{R_2+1} + C_1\}$, from the observation above we get (23). Use (H3)* to select a $\epsilon_0 > 0$ ($\epsilon_0 < 1$) such that $0 \leq u \leq p + \epsilon_0$ and $0 \leq t \leq 1$ implies

$$f(t, u) < \eta^* p.$$

Define

$$v_1(t) = -\eta^* p\left(\frac{1}{2}t^2 - \frac{\alpha(\gamma + 2\delta)}{2\rho}t - \frac{\beta(\gamma + 2\delta)}{2\rho}\right) + \epsilon_0.$$

Then a direct calculation shows that

$$\epsilon_0 \leq \min_{0 \leq t \leq 1} v_1(t) \leq \max_{0 \leq t \leq 1} v_1(t) = p + \epsilon_0,$$

and

$$-v_1''(t) = \eta^* p > f\big(t, v_1(t)\big), \qquad 0 < t < 1,$$
$$\begin{cases} \alpha v_1(0) - \beta v_1'(0) = \alpha\epsilon_0 \geq 0, \\ \gamma v_1(1) + \delta v_1'(1) = \gamma\epsilon_0 \geq 0, \end{cases}$$

therefore $v_1(t)$ is a strict supersolution of (1)–(2). Using the first inequality of (H1)*, we can find a $\delta > 0$ ($\delta < \epsilon_0$) such that for any $0 < u \leq \delta$ and $0 \leq t \leq 1$,

$$f(t, u) > \lambda_1 u.$$

Denote $u_1(t) = \delta\phi_1(t)$, then $u_1 < v_1$ and

$$-u_1''(t) = -\big(\delta\phi_1(t)\big)'' = \lambda_1 \delta\phi_1(t) < f\big(t, u_1(t)\big), \qquad 0 < t < 1,$$
$$\begin{cases} \alpha u_1(0) - \beta u_1'(0) = 0, \\ \gamma u_1(1) + \delta u_1'(1) = 0, \end{cases}$$

which means that $u_1(t)$ is a strict subsolution of (1)–(2). The conditions of Lemma 4 are satisfied for the BVP (1)–(2), hence Lemma 4 guarantees that (1)–(2) has a solution $x_1(t)$ satisfying

$$u_1 < x_1 < v_1. \tag{24}$$

We now construct a strict subsolution $u_2(t)$ of (22)–(2) satisfying $u_2 \nleq v_1$, and we will do this separately in different cases. The condition $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$ includes exactly eight cases: (i) $\alpha > 0$, $\beta = 0$, $\gamma = 0$, $\delta > 0$; (ii) $\alpha = 0$, $\beta > 0$, $\gamma > 0$, $\delta = 0$; (iii) $\alpha > 0$, $\beta = 0$, $\gamma > 0$, $\delta = 0$; (iv) $\alpha > 0$, $\beta = 0$, $\gamma > 0$, $\delta > 0$; (v) $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta = 0$; (vi) $\alpha > 0$, $\beta > 0$, $\gamma = 0$, $\delta > 0$; (vii) $\alpha = 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$; (viii) $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$.

Case (i). It is easy to see that $\lambda_1 = (\pi/2)^2$ and $\phi_1(t) = \sin(t\pi/2)$ in this case. For ϵ in (21), take a $c \in (0, 1/2)$ such that $(\pi/(2-2c))^2 < (\pi/2)^2 + \epsilon$, and define

$$u_2(t) = (R_2 + 1)\left(\left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(t-c)\pi}{2-2c} + 1\right),$$

then $u_2(0) = u_2'(1) = 0$ and $u_2(t) > 0$ for $0 < t \le 1$. If $0 \le t \le c$, we have that

$$\begin{aligned} -u_2''(t) &= (R_2 + 1)\left(\frac{\pi}{2-2c}\right)^2 \left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(t-c)\pi}{2-2c} \le 0 \\ &\le f_1(t, u_2(t)), \end{aligned}$$

and if $c < t \le 1$, from the definition of $f_1(t,u)$, we see that

$$\begin{aligned} -u_2''(t) &= (R_2 + 1)\left(\frac{\pi}{2-2c}\right)^2 \left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(t-c)\pi}{2-2c} \\ &\le (R_2 + 1)\left(\frac{\pi}{2-2c}\right)^2 \left(\left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(t-c)\pi}{2-2c} + 1\right) \\ &< (R_2 + 1)\left(\left(\frac{\pi}{2}\right)^2 + \epsilon\right) \left(\left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(t-c)\pi}{2-2c} + 1\right) \\ &= f_1(t, u_2(t)). \end{aligned}$$

Hence $u_2(t)$ is a strict subsolution of (22)–(2) and $u_2 \not\le v_1$.

Case (ii). In this case $\lambda_1 = (\pi/2)^2$ and $\phi_1(t) = \sin((t+1)\pi/2)$. Take $0 < c < 1/2$ such that $(\pi/(2-2c))^2 < (\pi/2)^2 + \epsilon$, and define

$$u_2(t) = (R_2 + 1)\left(\left(\sin\frac{c\pi}{2-2c}\right)^{-1} \sin\frac{(1-t-c)\pi}{2-2c} + 1\right),$$

then just as in Case (i), $u_2(t)$ is a strict subsolution of (22)–(2) and $u_2 \not\le v_1$.

Case (iii). In this case $\lambda_1 = \pi^2$ and $\phi_1(t) = \sin \pi t$. Fix a $0 < c < 1/4$ such that $(\pi/(1-2c))^2 < \pi^2 + \epsilon$, then just as in Case (i),

$$u_2(t) = (R_2 + 1)\left(\left(\sin\frac{c\pi}{1-2c}\right)^{-1} \sin\frac{(t-c)\pi}{1-2c} + 1\right)$$

is a strict solution of (22)–(2) and $u_2 \not\le v_1$.

Case (iv). In this case $\lambda_1 = (\pi/(1+\Delta))^2$ and $\phi_1(t) = \sin(t\pi/(1+\Delta))$, where $0 < \Delta < 1$ and $\pi/(1+\Delta)$ is the only solution in $(\pi/2, \pi)$ of the equation $\tan x = -\delta x/\gamma$. Let $0 < c < 1/3$ satisfy $(\pi/(1+\Delta)(1-c))^2 < (\pi/(1+\Delta))^2 + \epsilon$, and define

$$u_2(t) = M\left(\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{(t-c)\pi}{(1+\Delta)(1-c)} + (1-t)\right),$$

in which the number $M > 0$ is taken to be large enough that $\max_{c \le t \le 1} u_2(t) \ge R_2 + 1$. For $c > 0$ sufficiently small, it is easy to prove that $u_2(t) \geq 0$, for $0 \leq t \leq 1$. Fix such a c. If $0 \leq t \leq c$, then

$$\begin{aligned} -u_2''(t) &= M\left(\frac{\pi}{(1+\Delta)(1-c)}\right)^2 \left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \\ &\qquad \times \sin\frac{(t-c)\pi}{(1+\Delta)(1-c)} \\ &\leq 0 \leq f_1(t, u_2(t)). \end{aligned}$$

If $c < t \leq 1$, then

$$\begin{aligned} -u_2''(t) &\leq M\left(\frac{\pi}{(1+\Delta)(1-c)}\right)^2 \\ &\qquad \times \left(\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{(t-c)\pi}{(1+\Delta)(1-c)} + (1-t)\right) \\ &= \left(\frac{\pi}{(1+\Delta)(1-c)}\right)^2 u_2(t) \\ &< \left(\left(\frac{\pi}{1+\Delta}\right)^2 + \epsilon\right) u_2(t) \\ &= f_1(t, u_2(t)). \end{aligned}$$

On the other hand, $u_2(0) = 0$ and using $\tan(\pi/(1+\Delta)) = -\delta\pi/\gamma(1+\Delta)$, we see that

$$\begin{aligned} &\gamma u_2(1) + \delta u_2'(1) \\ &\quad = M\gamma\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{\pi}{1+\Delta} \\ &\qquad + M\delta\left(\frac{\pi}{(1+\Delta)(1-c)}\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \cos\frac{\pi}{1+\Delta} - 1\right) \\ &\quad = M\gamma\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{\pi}{1+\Delta} \\ &\qquad - \frac{\gamma M}{1-c}\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{\pi}{1+\Delta} - M\delta \\ &\quad = -\frac{c}{1-c} M\gamma\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1} \sin\frac{\pi}{1+\Delta} - M\delta < 0. \end{aligned}$$

Therefore $u_2(t)$ is a strict subsolution of (22)—(2) and $u_2 \nleq v_1$.

Case (v). In this case $\lambda_1 = (\pi/(1+\Delta))^2$ and $\phi_1(t) = \sin((t+\Delta)\pi/(1+\Delta))$, where $0 < \Delta < 1$ and $\pi/(1+\Delta)$ is the only solution in $(\pi/2, \pi)$ of the equation $\tan x = -\beta x/\alpha$. Take a $0 < c < 1/3$ such that $(\pi/(1+\Delta)(1-c))^2 < (\pi/(1+\Delta))^2 + \epsilon$, then just as Case (iv),

$$u_2(t) = M\left(\left(\sin\frac{c\pi}{(1+\Delta)(1-c)}\right)^{-1}\sin\frac{(1-t-c)\pi}{(1+\Delta)(1-c)} + t\right),$$

is a strict subsolution of (22)–(2) and $u_2 \nleq v_1$.

Case (vi). In this case $\lambda_1 = (\pi/(2+2\Delta))^2$ and $\phi_1(t) = \sin((t+\Delta)\pi/(2+2\Delta))$, where $0 < \Delta$ and $\pi/(2+2\Delta)$ is the only solution in $(0, \pi/2)$ of the equation $\cot x = \beta x/\alpha$.

Case (vii). In this case $\lambda_1 = (\pi/(2+2\Delta))^2$ and $\phi_1(t) = \sin((1-t+\Delta)\pi/(2+2\Delta))$, where $0 < \Delta$ and $\pi/(2+2\Delta)$ is the only solution in $(0, \pi/2)$ of the equation $\cot x = \delta x/\gamma$.

Case (viii). In this case $\lambda_1 = (\pi/(\Delta_1+\Delta_2))^2$ and $\phi_1(t) = \sin((t+\Delta_1)\pi/(\Delta_1+\Delta_2))$, where $0 < \Delta_1$, $1 < \Delta_2$, $0 < (\Delta_2 - \Delta_1)/2 < 1$, and satisfy

$$\alpha\sin\frac{\Delta_1\pi}{\Delta_1+\Delta_2} - \beta\frac{\pi}{\Delta_1+\Delta_2}\cos\frac{\Delta_1\pi}{\Delta_1+\Delta_2} = 0,$$

$$\gamma\sin\frac{(\Delta_1+1)\pi}{\Delta_1+\Delta_2} + \delta\frac{\pi}{\Delta_1+\Delta_2}\cos\frac{(\Delta_1+1)\pi}{\Delta_1+\Delta_2} = 0.$$

In the last three cases, $\phi_1(t) > 0$ for $0 \le t \le 1$, which guarantees that $M\phi_1(t) \ge R_2 + 1$ for $0 \le t \le 1$ if M is sufficiently large. Fix such a M and let

$$u_2(t) = M\phi_1(t),$$

then

$$-u_2''(t) = \lambda_1 M\phi_1(t) < (\lambda_1 + \epsilon)M\phi_1(t) = f_1(t, u_2(t)),$$

i.e., $u_2(t)$ is a strict subsolution of (22)–(2) and $u_2 \nleq v_1$.

To sum up, we have found a strict subsolution $u_2(t)$ of (22)–(2) with the property $u_2 \nleq v_1$ in all the eight cases. On the other hand, since $v_1(t)$ is a strict supersolution of (22)–(2), we have proved that all conditions in Lemma 5 are satisfied for the BVP (22)–(2), and therefore there exists a positive solution $x_2(t)$ of (22)–(2) satisfying

$$u_2 \nleq x_2 \nleq v_1. \tag{25}$$

By the a priori estimate mentioned above, we have $\|x_2\| < \tilde{R}_1 \leq R_2$, and therefore $x_2(t)$ is a positive solution of (1)–(2). From (24) and (25) we see that $x_1 \neq x_2$. The proof is completed.

Remark 3. A direct computation shows that $\eta^* = \eta$ if $\alpha\gamma = 0$ and $\eta^* > \eta$ if $\alpha\gamma > 0$.

Using arguments as in the proof of Theorems 1 and 2, we get the following two theorems.

THEOREM 4. *If*

$$\liminf_{u \to 0+} \min_{t \in [0,1]} \frac{f(t,u)}{u} > \lambda_1, \qquad \limsup_{a \to +\infty} \max_{t \in [0,1]} \frac{f(t,u)}{u} < \lambda_1,$$

then (1)–(2) *has at least one positive solution.*

THEOREM 5. *If*

$$\limsup_{u \to 0+} \max_{t \in [0,1]} \frac{f(t,u)}{u} < \lambda_1, \qquad \liminf_{u \to +\infty} \max_{t \in [0,1]} \frac{f(t,u)}{u} > \lambda_1,$$

then (1)–(2) *has at least one positive solution.*

Remark 4. From the point of view of variational methods and critical point theory, the number λ_1 in Theorems 4 and 5, and therefore in Theorems 1 and 3, is optimal.

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